Technical Notes

Computer Analysis of Asymmetrical Deformation of Orthotropic Shells of Revolution

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FINITE difference methods of digital computer solution of the field equations for elastic shells of revolution have been described previously for isotropic shells subject to sinusoidal loads[1] and for orthotropic shells subject to axisymmetric loads [2]† In the present paper a method of forward integration is developed for the more general case of orthotropic shells subject to sinusoidal loads The shell theory adopted for this purpose is that of Ref 4, except that the elasticity relations for isotropic, homogeneous shells [Eqs (10 8) of Ref 4] have been generalized to cover the case of orthotropic, heterogeneous shells These relations are

$$\left.\begin{aligned} T_1 &= C_1^{(0)}\epsilon_1 + C_{12}^{(0)}\epsilon_2 + C_1^{(1)}\kappa_1 + C_{12}^{(1)}\kappa_2 - \Theta_1^{(0)} \\ T_2 &= C_{12}^{(0)}\epsilon_1 + C_2^{(0)}\epsilon_2 + C_{12}^{(1)}\kappa_1 + C_2^{(1)}\kappa_2 - \Theta_2^{(0)} \\ M_1 &= C_1^{(1)}\epsilon_1 + C_{12}^{(1)}\epsilon_2 + C_1^{(2)}\kappa_1 + C_{12}^{(2)}\kappa_2 - \Theta_1^{(1)} \\ M_2 &= C_{12}^{(1)}\epsilon_1 + C_2^{(1)}\epsilon_2 + C_{12}^{(2)}\kappa_1 + C_2^{(2)}\kappa_2 - \Theta_2^{(1)} \\ T_{12} - M_{21}/R_2 &= T_{21} - M_{12}/R_1 = G^{(0)}\epsilon_{12} + 2G^{(1)}\kappa_{12} \\ (\tfrac{1}{2})(M_{12} + M_{21}) &= G^{(1)}\epsilon_{12} + 2G^{(2)}\kappa_{12} \end{aligned}\right\} \quad (1)$$

where

$$\left.\begin{aligned} C_1^{(\mu)} &= \int [E_1/(1 - \nu_1\nu_2)]z^\mu dz \\ C_2^{(\mu)} &= \int [E_2/(1 - \nu_1\nu_2)]z^\mu dz \\ C_{12}^{(\mu)} &= \int [\nu_1 E_2/(1 - \nu_1\nu_2)]z^\mu dz \\ &= \int [\nu_2 E_1/(1 - \nu_1\nu_2)]z^\mu dz \\ G^{(\mu)} &= \int E_{12} z^\mu dz \\ \Theta_1^{(\mu)} &= \int [E_1/(1 - \nu_1\nu_2)](\vartheta_1 + \nu_2\vartheta_2)z^\mu dz \\ \Theta_2^{(\mu)} &= \int [E_2/(1 - \nu_1\nu_2)](\vartheta_2 + \nu_1\vartheta_1)z^\mu dz \end{aligned}\right\} \quad (2)$$

where E_1, E_2, E_{12} are Young s moduli and the shear modulus, ν_1, ν_2 are Poisson's ratios, and ϑ_1, ϑ_2 are free thermal strains To simplify Eqs (1) the reference surface $z = 0$ is chosen so that‡

$$C_{12}^{(1)} = 0 \quad (3)$$

For machine integration it is convenient to express the shell equations in terms of the same generalized force and displacement variables by which the boundary conditions are naturally expressed If these variables are referred to axial, radial (with respect to circles of latitude), and circumferential directions, instead of the conventional meridional, normal, and circumferential directions, the resulting differential equations are free of the meridional curvature as well as meridional derivatives of wall rigidities With the notation shown in Fig 1, static and kinematic relations become (in terms of harmonic amplitudes of load and response variables)

$$\left.\begin{aligned} P' &= -(r'/r)P - (n/R_2)S - n^2(r'/r^2)M_2 + (n/r^2)(M_{12} + M_{21}) - (r/R_2)X_1 + r' X_3 - n(r'/r)L_1 \\ Q' &= -(r'/r)Q - n(r'/r)S + (1/r)T_2 + (n^2/rR_2)M_2 - r'X_1 - (r/R_2)X_3 + (n/R_2)L_1 \\ S' &= -2(r'/r)S + (n/r)T_2 + (n/rR_2)M_2 - X_2 + (1/R_2)L_1 \\ M_1' &= -r'P + (r/R_2)Q - (r'/r)M_1 + (r'/r)M_2 - (n/r)(M_{12} + M_{21}) - L_2 \\ \xi' &= r'\chi + (r/R_2)\epsilon_1 \\ \eta' &= -(r/R_2)\chi + r'\epsilon_1 \\ v' &= (n/R_2)\xi + n(r'/r)\eta + (r'/r)v + \epsilon_{12} \\ \chi' &= \kappa_1 \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} T_1 &= (r/R\ )P + r'Q \\ \epsilon_2 &= (\eta + nv)/r \\ \kappa_2 &= [-n^2(r'/r)\xi + (n^2/R_2)\eta + (n/R_2)v + r'\chi]/r \\ \kappa_{12} - \epsilon_{12}/R_2 &= (n/r)(\xi/r - \chi) \end{aligned}\right\} \quad (5)$$

where $(\ )' = d(\ )/ds$, s being meridional arc length, X_1, X_2, X_3 and L_1, L_2 indicate applied surface forces and moments per unit area, respectively, and n is the harmonic number In addition, the following partially inverted forms of the elasticity relations (1) are used:

$$\left.\begin{aligned} T_2 &= \lambda_{11}\epsilon_2 + \lambda_{12}\kappa_2 + \lambda_{13}T_1 + \lambda_{14}M_1 + \lambda_{13}\Theta_1^{(0)} + \lambda_{14}\Theta_1^{(1)} - \Theta_2^{(0)} \\ M_2 &= \lambda_{21}\epsilon_2 + \lambda_{22}\kappa_2 + \lambda_{23}T_1 + \lambda_{24}M_1 + \lambda_{23}\Theta_1^{(0)} + \lambda_{24}\Theta_1^{(1)} - \Theta_2^{(1)} \\ \epsilon_1 &= \lambda_{31}\epsilon_2 + \lambda_{32}\kappa_2 + \lambda_{33}T_1 + \lambda_{34}M_1 + \lambda_{33}\Theta_1^{(0)} + \lambda_{34}\Theta_1^{(1)} \\ \kappa_1 &= \lambda_{41}\epsilon_2 + \lambda_{42}\kappa_2 + \lambda_{43}T_1 + \lambda_{44}M_1 + \lambda_{43}\Theta_1^{(0)} + \lambda_{44}\Theta_1^{(1)} \\ M_{12} + M_{21} &= \mu_{11}(\kappa_{12} - \epsilon_{12}/R_2) + \mu_{12}S \\ \epsilon_{12} &= \mu_{21}(\kappa_{12} - \epsilon_{12}/R_2) + \mu_{22}S \end{aligned}\right\} \quad (6)$$

where

$$\left.\begin{aligned} \lambda_{11} &= C_2^{(0)} - C_1^{(2)}C_{12}^{(0)2}/\Delta \\ \lambda_{12} &= \lambda_{21} = C_2^{(1)} + C_1^{(1)}C_{12}^{(0)}C_{12}^{(2)}/\Delta \\ \lambda_{13} &= -\lambda_{31} = C_{12}^{(0)}C_1^{(2)}/\Delta \\ \lambda_{14} &= -\lambda_{41} = -C_{12}^{(0)}C_1^{(1)}/\Delta \\ \lambda_{22} &= C_2^{(2)} - C_1^{(0)}C_{12}^{(2)2}/\Delta \\ \lambda_{23} &= -\lambda_{32} = -C_{12}^{(2)}C_1^{(1)}/\Delta \\ \lambda_{24} &= -\lambda_{42} = C_{12}^{(2)}C_1^{(0)}/\Delta \\ \lambda_{33} &= C_1^{(2)}/\Delta \\ \lambda_{34} &= \lambda_{43} = -C_1^{(1)}/\Delta \\ \lambda_{44} &= C_1^{(0)}/\Delta \\ \Delta &= C_1^{(0)}C_1^{(2)} - C_1^{(1)2} \\ \mu_{11} &= 4(G^{(0)}G^{(2)} - G^{(1)2})\mu_{22} \\ \mu_{12} &= -\mu_{21} = 2[G^{(1)} + (2/R_2)G^{(2)}]\mu_{22} \\ \mu_{22} &= [G^{(0)} + (4/R_2)G^{(1)} + (2/R_2)^2G^{(2)}]^{-1} \end{aligned}\right\} \quad (7)$$

Equations (4), supplemented by Eqs (5) and (6), are a system of first-order differential equations that may be written conveniently in vector form as§

$$\mathbf{Y}' = \mathbf{F}(\mathbf{Y}, s) \quad (8)$$

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† A discrete element method for general shells is presented in Ref 3

‡ For positive Poisson's ratios, the reference surface lies within the shell wall, so that for practical purposes its geometric properties are the same as those of any convenient surface within the shell wall

§ The explicit form of Eq (8) is not needed, as it is more convenient in machine integration to use Eqs (4–6) directly

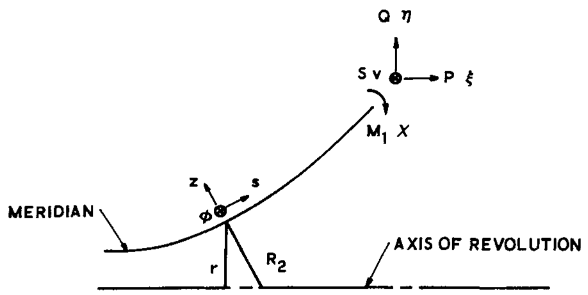

Fig 1 Axial section of shell of revolution

Method of Solution of Eq (8)

The range of s is divided into m subintervals Points of subdivision are chosen corresponding to interior rings and other meridional discontinuities The Runge-Kutta method of forward integration is used to obtain over each subinterval a set of eight independent complementary solution vectors $\mathbf{Y}_i^{(k)}(s)$ ($k = 1, \quad , 8$; $i = 1, \quad , m$) and a particular solution vector $\mathbf{Y}_i^{(9)}(s)$ of Eq (8) Initially, at the starting point of each subinterval, the matrix of column vectors $[\mathbf{Y}_i^{(k)}]$ is the identity matrix, and $\mathbf{Y}_i^{(9)}$ is the null vector Additional points of subdivision are, in general, necessary to insure that the components of the vectors $\mathbf{Y}_i^{(k)}$, $\mathbf{Y}_i^{(9)}$ do not grow so large that numerical significance will be lost in their superposition These solutions are retained at each designated output point and at the final point of each subinterval in the form of four 4×4 matrices U_i, V_i, W_i, Z_i and two four-element column vectors $\mathbf{G}_i$, $\mathbf{J}_i$, which are defined by partitioning $[\mathbf{Y}_i^{(k)}]$ and $\mathbf{Y}_i^{(9)}$ in the following manner¶:

$$[\mathbf{Y}_i^{(k)}] = \left[\begin{array}{c:c} U_i & V_i \\ \hdashline W_i & Z_i \end{array}\right] \qquad \mathbf{Y}_i^{(9)} = \left\{\begin{array}{c} \mathbf{G}_i \\ \hdashline \mathbf{J}_i \end{array}\right\} \tag{9}$$

If the desired solution vectors $\mathbf{Y}_i$ and the vectors $\mathbf{C}_i$ of a suitable set of superposition constants C_i^k (for the $\mathbf{Y}_i^{(k)}$) are each partitioned into two four-element vectois according to

$$\mathbf{Y}_i = \left\{\begin{array}{c} \mathbf{y}_i \\ \hdashline \mathbf{z}_i \end{array}\right\} \qquad \mathbf{C}_i = \left\{\begin{array}{c} \mathbf{c}_i \\ \hdashline \mathbf{d}_i \end{array}\right\} \tag{10}$$

then the desired solution may be written as

$$\begin{array}{l} \mathbf{y}_i = U_i\,\mathbf{c}_i + V_i\,\mathbf{d}_i + \mathbf{G}_i \\ \mathbf{z}_i = W_i\,\mathbf{c}_i + Z_i\,\mathbf{d}_i + \mathbf{J}_i \end{array} \qquad i = 1, \quad , m \tag{11}$$

The problem is thus reduced to one of finding $\mathbf{c}_i$, $\mathbf{d}_i$, so that Eqs (11) can be applied The equations that determine these constants come from the edge conditions and interior conditions at the points of subdivision Edge conditions may be written as**

$$\begin{array}{l} B_0\,\mathbf{y}_{1_0} + D_0\,\mathbf{z}_{1_0} = \mathbf{L}_0 \\ B_m\,\mathbf{y}_{m_1} + D_m\,\mathbf{z}_{m_1} = \mathbf{L}_n \end{array} \tag{12}$$

and interior conditions may be written as

$$\begin{array}{l} B_i^+\mathbf{y}_{i+1_0} + B_i^-\mathbf{y}_{i_1} + D_i\,\mathbf{z}_{i_1} = \mathbf{L}_i \\ \mathbf{z}_{i_1} = \mathbf{z}_{i+1_0} \end{array} \qquad i = 1, \quad , m - 1 \tag{13}$$

where the second-order subscript zero or one is used to indicate evaluation at the initial point or the final point, respectively, of the subinterval indicated by the first-order subscript From Eqs (11–13), the following tri-diagonal system of $2m$ vector equations may be derived:

$$\left.\begin{array}{rl} D_0\,\mathbf{d}_1 + B_0\,\mathbf{c}_1 = \mathbf{L}_0 & (14a) \\ -Z_{i_1}\,\mathbf{d}_i - W_{i_1}\,\mathbf{c}_i + \mathbf{d}_{i+1} = \mathbf{J}_{i_1} & (14b) \\ \gamma_i\,\mathbf{c}_i + \delta_i\,\mathbf{d}_{i+1} + B_i^+\,\mathbf{c}_{i+1} = \mathbf{\Omega}_i & (14c) \\ \beta\,\mathbf{d}_m + \alpha\mathbf{c}_m = \mathbf{\Psi} & (14d) \end{array}\right\} \quad i = 1, \quad , m - 1$$

where

$$\left.\begin{array}{l} \gamma_i = B_i^-\,U_{i_1} + \Delta_i\,W_{i_1} \\ \delta_i = D_i - \Delta_i \\ \mathbf{\Omega}_i = \mathbf{L}_i - B_i^-\,\mathbf{G}_{i_1} - \Delta_i\,\mathbf{J}_{i_1} \\ \Delta_i = -B_i^-\,V_{i_1}\,Z_{i_1}^{-1} \\ \alpha = B_m\,U_{m_1} + D_m\,W_{m_1} \\ \beta = B_m\,V_{m_1} + D_m\,Z_{m_1} \\ \mathbf{\Psi} = \mathbf{L}_m - B_m\,\mathbf{G}_{m_1} - D_m\,\mathbf{J}_{m_1} \end{array}\right\} \tag{15}$$

Solution of Eqs (14) by Gaussian elimination yields††

$$\left.\begin{array}{lll} \mathbf{c}_m = (\alpha - \beta p_m')^{-1}(\mathbf{\Psi} - \beta\mathbf{q}_m') & & \\ \mathbf{d}_i = -p_i'\,\mathbf{c}_i + \mathbf{q}_i' & i = 2, & , m \\ \mathbf{c}_i = p_i\,\mathbf{d}_{i+1} - \mathbf{q}_i & i = 1, & , m - 1 \\ \mathbf{d}_1 = Z_{1_1}^{-1}(\mathbf{d}_2 - W_{1_1}\mathbf{c}_1 - \mathbf{J}_{1_1}) & & \end{array}\right\} \tag{16}$$

where

$$\left.\begin{array}{ll} p_1 = (D_0 Z_{1_1}^{-1} W_{1_1} - B_0)^{-1} D_0 Z_{1_0}^{-1} & \\ \mathbf{q}_1 = (D_0 Z_{1_1}^{-1} W_{1_1} - B_0)^{-1}(D_0 Z_{1_0}^{-1}\mathbf{J}_{1_1} + \mathbf{L}_0) & \\ p_{i+1}' = (\delta_i + \gamma_i p_i)^{-1} B_i^+ & i = 1, \quad , m - 1 \\ \mathbf{q}_{i+1}' = (\delta_i + \gamma_i p_i)^{-1}(\Omega_i + \gamma_i \mathbf{q}_i) & i = 1, \quad , m - 1 \\ p_i = (W_{i_1} - Z_{i_1} p_i')^{-1} & i = 2, \quad m - 1 \\ \mathbf{q}_i = (W_{i_1} - Z_{i_1} p_i')^{-1}(\mathbf{J}_{i_1} + Z_{i_1}\mathbf{q}_i') & i = 2, \quad m - 1 \end{array}\right\} \tag{17}$$

The case $m = 1$ is a special one that must be treated separately, since for this case there are no interior points of

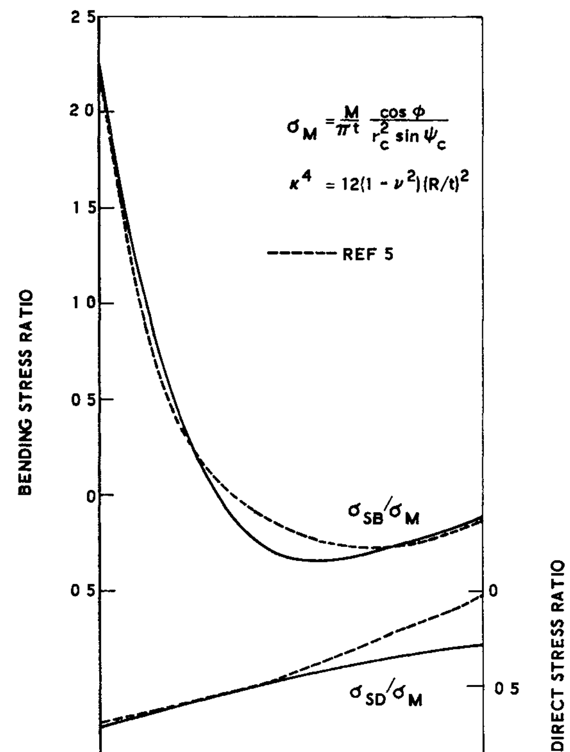

Fig 2 Stiesses due to tilting moment M

¶ These submatrices satisfy the following reciprocal relations: $U_i^T W_i = W_i^T U_i$, $V_i^T Z_i = Z_i^T V_i$, and $U_i^T Z_i - W_i^T V_i = (r_{i_0}/r_i) I$, where the superscript T denotes transpose, r_{i_0} is the initial value of r_i, and I is the identity matrix

** In the remainder of this discussion all lightface symbols represent 4×4 matrices, and all boldface symbols represent four-element column vectors

†† Primed symbols in Eqs (16) and (17) are not to be confused with derivatives of unprimed symbols but simply denote different matrices from the corresponding unprimed matrices

subdivision, and hence Eqs (14b) and (14c) do not apply The edge conditions supply the only applicable equations, viz , Eqs (14a) and (14d) Their solution may be written as

$$\mathbf{d}_1 = (B_0\alpha^{-1}\beta - D_0)^{-1}(B_0\alpha^{-1}\mathbf{\Psi}^* - \mathbf{L}_0)$$
$$\mathbf{c}_1 = \alpha^{-1}(\mathbf{\Psi}^* - \beta\mathbf{d}_1) \tag{18}$$

After calculation of the forces and displacements $\mathbf{y}_i$, $\mathbf{z}_i$ by means of Eqs (11), shell stresses may be obtained by application of Eqs (5) and (6) (for the strains) and Hooke's law Transverse shear stresses, which may be important in bending regions,‡‡ may be obtained by application of, in addition, Eqs (4) and derivatives of Eqs (5) and (6) (for the strain derivatives) and of Hooke's law and integration of the three-dimensional equilibrium equations (with suitable simplifications)

Example

Figures 2 and 3 show meridional bending and direct stress distributions in the vicinity of the small edge of a zone of a spherical shell for which the colatitude ψ ranges from 0 4668 to $\pi/2$ rad, $R/t = 50$, and $\nu = 0\ 3$ (isotropic) under the action of a tilting moment M and a lateral thrust H applied through a rigid ring at the edge $\psi = \psi = 0\ 4668$ These numerical results are compared to the analytical solutions for the same problems obtained by Steele,[5] who uses the same definitions of shell strains as in the present paper§§ but assumes Love's first approximation for the elasticity relations Excellent agreement is seen at the edge, where the maximum stress conditions occurs The relatively small discrepancy in the interior apparently results from simplifications (based on order-of-magnitude considerations) made in Ref 5 in order to arrive at analytical results

In order to determine the effect of meridional and circumferential stiffening, these problems were repeated for an orthotropic material for which $\nu_1 = 0\ 3$, $E_1/E_2 = 2$, and $E_1/E_{12} = 5\ 2$, oriented so that the maximum stiffness direction is firstly meridional and secondly circumferential The results show that this degree of orthotropy is not great enough

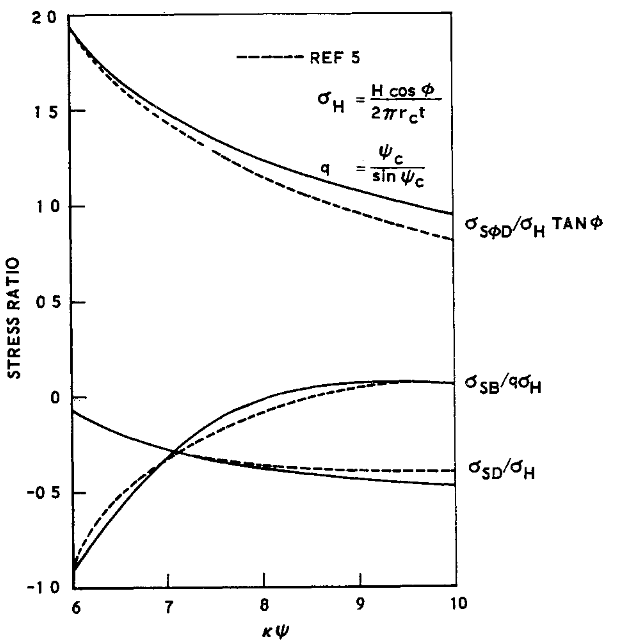

Fig 3 **Stresses due to lateral thrust H**

‡‡ The transverse normal stress need not be computed, since it is negligible even in bending regions

§§ The only disagreement occurs in the last term of his expression for κ_{12} [Eq (4f) of Ref 5], which is apparently a misprint since it is dimensionally incorrect

Table 1 Percentage changes in the maximum bending stresses

E/E_ϕ	M, %	H, %
2	+21	+79
$\frac{1}{2}$	−9 5	+58

to alter appreciably the direct stresses (i e , the membrane stresses are still, to a good approximation, uncoupled from the bending stresses) The percentage changes in the maximum bending stresses from those of the isotropic cases are shown in Table 1

It is noted that the six problems compared in Table 1 took a total of 6 min of Philco 2000 computer time and that two interior points of subdivision were sufficient

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Potential Flow Past a Parabolic Leading Edge

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Nomenclature

a,b	=	constants
h,s	=	coordinates of singular point
x,y,ξ,η,ξ',η'	=	rectangular coordinates
z	=	$x + iy$
ζ	=	$\xi + i\eta$
ζ'	=	$\xi' + i\eta'$
Q	=	complex potential
r	=	leading-edge radius
c	=	chord
C_p	=	pressure coefficient, $1 - [U/U_\infty]^2$
U,U_∞	=	local and remote flow speeds, respectively

Subscripts

stag	=	stagnation point
$C_{p_{\min}}$	=	minimum-pressure point

Study of an exact solution for the potential flow past a parabolic cylinder has led to insight into the relation between the locations of maximum and minimum pressure on an airfoil The following rule may be used to locate either of these two points when the other is known: a straight line from the stagnation point to the minimum-pressure point will always pass through a fixed third point that lies midway between the airfoil leading edge and the leading-edge center of curvature (see Fig 1)

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